DR. MATHIAS BRAUN & WENHAO ZHAO

FUNCTIONAL ANALYSIS 2
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SHEET 3

Exercises for 'Functional Analysis 2' [MATH-404]

(03/03/2025)

Ex 3.1 (On sequences and boundedness in LCTVS)
Let X be a LCTVS with its topology being induced by a family of seminorms $(p_i)_{i\in I}$.

a) Show that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ if and only if $p_i(x_n - x) \to 0$ for all $i \in I$.

A set $E \subset X$ is called bounded if for every neighborhood U of 0 there exists $s > 0$ such that $E \subset sU$.

b) Show that a set $E \subset X$ is bounded if and only if $p_i(E)$ is a bounded subset of $\mathbb{R}$ for every $i \in I$.

Recall that a sequence $(x_n)_{n\in N} \subset X$ is called a **Cauchy sequence** if for every neighborhood U of 0 there exists $N \in \mathbb{N}$ such that $x_m - x_n \in U$ for all $n, m \geq N$.

c) Show that Cauchy sequences are bounded.
Hint: If U is a neighborhood of the origin, due to the continuity of the addition and Ex. 1.3, there exists an absorbing, balanced neighborhood V of 0 such that $V + V \subset U$.

Ex 3.2 (On bounded and compact sets in TVS)
Let X be a TVS. Show that :

a) Every compact set $K \subset X$ is bounded.
Hint: Every neighborhood of 0 contains an absorbing and balanced neighborhood of 0, see Exercise 1.3 b)-c).

b) A set $E \subset X$ is bounded if and only if every countable subset of E is bounded.

c) If $E, F \subset X$ are bounded, so is $E + F$. If $E, F \subset X$ are compact, so is $E + F$.

d)* If $K \subset X$ is compact and $C \subset X$ is closed then $K + C$ is closed. Give an example of two closed sets in a TVS such that their sum is not closed.
Hint: Prove first that if K is contained in an open set U then there is an open neighborhood of 0 such that $K + V \subset U$.

Ex 3.3 (On the continuity of seminorms on LCTVS)
Let X be a locally convex topological vector space with seminorms $(p_i)_{i\in I}$ generating the topology. Consider another seminorm $q : X \to [0, +\infty)$. Show that the following properties are equivalent :

i) q is continuous.

ii) There exist $c > 0$ and $I_0 \subset I$ finite such that

$$q(x) \leq c \sum_{i\in I_0} p_i(x).$$

Ex 3.4 (On test functions on compact intervals*)
Let $[a, b]$, where $a < b$, be a compact interval in $\mathbb{R}$. Consider the vector space

$$\mathcal{D}_{[a,b]} = \{f \in C^\infty(\mathbb{R}) : \ \mathrm{supp}(f) \subseteq [a, b]\},$$

where $C^\infty(\mathbb{R})$ is the space of all smooth functions on $\mathbb{R}$ and $\mathrm{supp}(f)$ is the **support** of f (namely, the complement of the largest open set on which f vanishes).

a) Show that the function

$$\phi(x) = \begin{cases} e^{-1/t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

belongs to $C^\infty(\mathbb{R})$. As a consequence, notice that $f(x) = \phi(x - a)\phi(b - x)$ is in $\mathcal{D}_{[a,b]}$.

b) Consider the family of seminorms

$$p_n(f) = \max\left\{\left|f^{(k)}(x)\right| : \ x \in \mathbb{R}, \ k \leq n\right\}, \quad f \in \mathcal{D}_{[a,b]}, \ n = 0, 1, \dots,$$

where $f^{(k)}$ stands for k-th derivative of f. Show that this family introduces a locally convex topology on $\mathcal{D}_{[a,b]}$. What is the neighborhood basis of 0 in this topology. What does it mean that a sequence $\{f_n\} \subset \mathcal{D}_{[a,b]}$ converges to $f \in \mathcal{D}_{[a,b]}$? Is this topology metrizable/normable ?

c) Let $E \subset \mathcal{D}_{[a,b]}$ be closed and bounded in the topology from b). Show that for every $k = 0, 1, \dots,$ the set $\{f^{(k)} : \ f \in E\}$ is a precompact subset of $C([a, b])$ (the space of continuous functions on $[a, b]$ with the supremum norm). Using this fact, demonstrate that E is compact in $\mathcal{D}_{[a,b]}$.

Hint: For the first part use Arzelà–Ascoli theorem, for the second Cantor's diagonal argument. *(Let us know if you are not familiar with these tools)*